# ある平均値積分(III)
# A Mean Value Integral(III)

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**概 要**

**Abstract**

This paper is the continuation of the author's [1] [2].
We show that de la Valée Poussin's zero free region of the Riemann Zeta-Function follows as corrolary from our result. Also we consider the situation of our result in case of $\rho_0 = 1$.
Key words ; the Riemann zeta-function.
Mathematics Subject Classification 2010;　11M06.

$\zeta(s), (s = \sigma + it)$ を Riemann の zeta 関数とする。又, $\rho = \beta + i\gamma$ で $\zeta(s)$ の複素零点complex zero を表わすことにする。又, $\Gamma(w)$ は Euler の gamma 関数である。記号等は [1] に従う。$\gamma(a,z)$, $\Gamma(a,z)$, $P(a,z)$, $Q(a,z)$ を次で定義する。$a > 0$, $z > 0$ に対して

$$\gamma(a,z) := \int_0^z t^{a-1}e^{-t}dt, \quad P(a,z) := \frac{\gamma(a,z)}{\Gamma(a)}$$

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt, \quad Q(a,z) := \frac{\Gamma(a,z)}{\Gamma(a)}.$$

すると次の積分表示が得られる [4]。

$$Q(a,z) = -\frac{e^{-a\phi(\lambda)}}{2\pi i}\int_{c-i\infty}^{c+\infty} e^{a\phi(t)}\frac{dt}{t-\lambda}, \quad 0 < c < \lambda$$

$$P(a,z) = 1 + \frac{e^{-a\phi(\lambda)}}{2\pi i}\int_{c-i\infty}^{c+\infty} e^{a\phi(t)}\frac{dt}{t-\lambda},\quad 0<c<\lambda$$
$$where\quad \phi(t) := t - 1 - \log t,\quad \lambda := \frac{z}{a}$$

ここで，この積分の最急降下路 path of steepest descent

$$L := \{t := \rho e^{i\theta} \in \mathbf{C} \mid \rho := \frac{\theta}{\sin\theta},\ -\pi < \theta < \pi\}$$

又この積分の鞍点 saddle point は只一つで $t = 1$ である。
Temme [4] は，この最急降下路を使って，初めて $P(a,z)$，$Q(a,z)$ の次の漸近展開を得た: $a > 0$，$z > 0$ に対して

$$Q(a,z) = \frac{1}{2}\mathrm{erfc}(\eta\sqrt{\frac{a}{2}}) + R_a(\eta),$$
$$P(a,z) = \frac{1}{2}\mathrm{erfc}(-\eta\sqrt{\frac{a}{2}}) - R_a(\eta),$$
$$\eta := (\lambda-1)\sqrt{2\frac{\lambda-1-\log\lambda}{(\lambda-1)^2}},\quad \lambda := \frac{z}{a},$$
$$R_a(\eta) := \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}}\sum_{n=0}^{\infty}\frac{c_n(\eta)}{a^n},\quad a\to\infty,$$
$$\mathrm{erfc}(z) := \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}dt.$$

ここで

$$\eta = \begin{cases} +\sqrt{2(\lambda-1-\log\lambda)}, & (1<\lambda) \\ 0, & (\lambda=1) \\ -\sqrt{2(\lambda-1-\log\lambda)}, & (0<\lambda<1) \end{cases}$$

である。
これらの特別な場合として, $\lambda = 1$ として

$$\begin{aligned} P(a,a) &= \frac{1}{2}\mathrm{erfc}(0) - R_a(0) \\ &= \frac{1}{2} - \frac{1}{\sqrt{2\pi a}}\sum_{n=0}^{\infty}\frac{c_n(0)}{a^n},\quad a\to\infty \cdots\cdots (1) \end{aligned}$$

を得る。
Temme [4] の方法とは異なり，素直に最急降下法 method of steepest descent 或いは鞍点法 saddle point method を用いれば，又 (1) を用いれば次を得る [3]:

**補題 1**　$a \to \infty$ のとき，

$$P(a,\lambda a)=\begin{cases}\frac{e^{-a\phi(\lambda)}}{2\pi i}\int_L e^{a\phi(t)}\frac{dt}{t-\lambda}=\frac{e^{-a\phi(\lambda)}}{\sqrt{2\pi a}}\left[\frac{1}{1-\lambda}+O(\frac{1}{a})\right], & (0<\lambda<1)\\ \frac{1}{2}+O(\frac{1}{\sqrt{a}}), & (\lambda=1)\\ 1+\frac{e^{-a\phi(\lambda)}}{2\pi i}\int_L e^{a\phi(t)}\frac{dt}{t-\lambda}=1-\frac{e^{-a\phi(\lambda)}}{\sqrt{2\pi a}}\left[\frac{1}{\lambda-1}+O(\frac{1}{a})\right], & (1<\lambda)\end{cases}$$

証明 [3] を見よ。

**補題 2**　$\sigma>1$, $\nu\in\mathbf{N}$, $\nu\to\infty$ のとき，

$$\sum_{n\le\exp[\frac{\lambda(\nu+1)}{\sigma-1}]}\frac{\Lambda(n)(\log n)^{\nu-1}}{n^\sigma}\asymp\sum_{n\le\exp[\frac{\lambda(\nu+1)}{\sigma-1}]}\frac{(\log n)^\nu}{n^\sigma}$$
$$\asymp\left(\frac{1}{\sigma-1}\right)^{\nu+1}\Gamma(\nu+1)P(\nu+1,\lambda(\nu+1))$$

証明 素数定理 Prime Number Theorem より

$$\sum_{n\le\exp[\frac{\lambda(\nu+1)}{\sigma-1}]}\frac{\Lambda(n)(\log n)^{\nu-1}}{n^\sigma}\asymp\sum_{n\le\exp[\frac{\lambda(\nu+1)}{\sigma-1}]}\frac{(\log n)^\nu}{n^\sigma}$$

である。又 Euler-Maclaurin の総和公式より

$$\begin{aligned}&\sum_{n\le\exp[\frac{\lambda(\nu+1)}{\sigma-1}]}\frac{(\log n)^\nu}{n^\sigma}\\ \asymp&\int_1^{\exp[\frac{\lambda(\nu+1)}{\sigma-1}]}\frac{(\log t)^\nu}{t^\sigma}dt=\int_0^{\frac{\lambda(\nu+1)}{\sigma-1}}y^\nu e^{-(\sigma-1)y}dy\\ =&\left(\frac{1}{\sigma-1}\right)^{\nu+1}\int_0^{\lambda(\nu+1)}x^{(\nu+1)-1}e^{-x}dx\\ =&\left(\frac{1}{\sigma-1}\right)^{\nu+1}\gamma(\nu+1,\lambda(\nu+1))\\ =&\left(\frac{1}{\sigma-1}\right)^{\nu+1}\Gamma(\nu+1)P(\nu+1,\lambda(\nu+1))\end{aligned}$$

となり補題は証明された。□
この論文 (III) では $1<XY\ll 1$ の場合と $\rho_0=1$ の場合を扱う。
[1] の (7) より次の補題が得られる。

**補題 3**

$$-\frac{1+o(1)}{(\sigma-\beta_0)^{\nu+1}}\left\{\alpha+\frac{1}{2\pi i}F(X)\right\}$$

$$= \frac{1}{\nu!} \sum_{n \le (XY)^{\nu}} \frac{\Lambda(n)(\log n)^{\nu}}{n^{s}} \alpha_n +$$
$$+\mathrm{O}\left\{\left(\frac{Y}{\sigma}X\right)^{\nu}\right\} + \mathrm{O}\left\{\left(Y\frac{X}{U_m}\right)^{\nu}\right\}$$
$$with\ \alpha + \frac{1}{2\pi i}F(X) \neq 0,\ X := \exp\left[\frac{a(\nu+1)}{\nu(\sigma-c)}\right],\ Y := \exp\left[\frac{b(\nu+1)}{\nu(\sigma-c)}\right],$$
$$a, b > 0,\ c = \frac{1}{2} \text{ or } 1.$$
$$\cdots(2)$$

注 (2) は $\sigma \ge 1$ でも成立する。又ここで

$$\lim_{\nu\to\infty} X = \lim_{\nu\to\infty} \exp\left[\frac{a(\nu+1)}{\nu(\sigma-c)}\right] = \exp\left[\frac{a}{(\sigma-c)}\right]$$

であるので, $\nu \gg 1$ に対しては

$$\alpha + \frac{1}{2\pi i}F(X) \neq 0$$

は保たれる。
又同じく [1] の (9) より次の補題が得られる。

**補題 4**

$$\frac{1}{2h}\int_{\gamma_0-h}^{\gamma_0+h}\left|\frac{1+o(1)}{(\sigma-\beta_0)^{\nu+1}}\left\{\alpha+\frac{1}{2\pi i}F(X)\right\}\right|^{\frac{1}{\nu}} dt$$
$$\le \left[\mathrm{O}\left\{\frac{1}{2h}\int_{\gamma_0-h}^{\gamma_0+h}\left|\frac{1}{\nu!}\sum_{n\le(XY)^{\nu}}\frac{\Lambda(n)(\log n)^{\nu}}{n^{s}}\alpha_n\right|^2 dt+\right.\right.$$
$$\left.\left.+\left(Y\frac{X}{U_m}\right)^{2\nu}+\left(\frac{Y}{\sigma}X\right)^{2\nu}\right\}\right]^{\frac{1}{2\nu}}$$
$$= \left[\mathrm{O}\left\{\left(\frac{1}{\nu!}\right)^2\sum_{n\le(XY)^{\nu}}\frac{\Lambda(n)^2}{n^{2\sigma}}(\log n)^{2\nu}{\alpha_n}^2+\right.\right.$$
$$+\mathrm{O}\left(\frac{1}{h}\left(\frac{1}{\nu!}\right)^2\sum_{n\le(XY)^{\nu}}\frac{\Lambda(n)^2}{n^{2\sigma-1}}(\log n)^{2\nu}{\alpha_n}^2\right)+$$
$$\left.\left.+\left(Y\frac{X}{U_m}\right)^{2\nu}+\left(\frac{Y}{\sigma}X\right)^{2\nu}\right\}\right]^{\frac{1}{2\nu}}\cdots(3)$$

$$
\begin{aligned}
&= \left[ \mathrm{O}\left\{ \left(\frac{1}{\nu!}\right)^2 \sum_{n\le(XY)^\nu} \frac{\Lambda(n)^2}{n^{2\sigma}} (\log n)^{2\nu} {\alpha_n}^2 + \right.\right. \\
&+\mathrm{O}\left( \frac{(XY)^\nu}{h} \left(\frac{1}{\nu!}\right)^2 \sum_{n\le(XY)^\nu} \frac{\Lambda(n)^2}{n^{2\sigma}} (\log n)^{2\nu} {\alpha_n}^2 \right) + \\
&\left.\left. + \left(Y\frac{X}{U_m}\right)^{2\nu} + \left(\frac{Y}{\sigma}X\right)^{2\nu} \right\}\right]^{\frac{1}{2\nu}} \cdots (4) \\
&= \left[ \mathrm{O}\left\{ \frac{(XY)^\nu}{h} \left(\frac{1}{\nu!}\right)^2 \sum_{n\le(XY)^\nu} \frac{\Lambda(n)^2}{n^{2\sigma}} (\log n)^{2\nu} {\alpha_n}^2 + \right.\right. \\
&\left.\left. + \left(Y\frac{X}{U_m}\right)^{2\nu} + \left(\frac{Y}{\sigma}X\right)^{2\nu} \right\}\right]^{\frac{1}{2\nu}} \\
&\textit{with } \alpha + \frac{1}{2\pi i}F(X) \neq 0,\ X := \exp\left[\frac{a(\nu+1)}{\nu(\sigma-c)}\right],\ Y := \exp\left[\frac{b(\nu+1)}{\nu(\sigma-c)}\right], \\
&\quad a, b > 0,\ c = \frac{1}{2} \text{ or } 1. \cdots (5)
\end{aligned}
$$

注 (3),(4),(5) も $\sigma \geq 1$ でも成立する。又同じく

$$
\lim_{\nu\to\infty} X = \lim_{\nu\to\infty} \exp\left[\frac{a(\nu+1)}{\nu(\sigma-1)}\right] = \exp\left[\frac{a}{(\sigma-1)}\right]
$$

であるので, $\nu \gg 1$ に対しては

$$
\alpha + \frac{1}{2\pi i}F(X) \neq 0
$$

は保たれる。

**補題 5**　$\sigma > 1,\ \nu \in \mathbf{N},\ \nu \to \infty$ のとき,

$$
\sum_{n\le\exp[\frac{\lambda(\nu+1)}{\sigma-1}]} \frac{\Lambda(n)(\log n)^{\nu-1}}{n^\sigma}
\asymp \begin{cases}
\left(\frac{1}{\sigma-1}\right)^{\nu+1} \nu! \frac{e^{-(\nu+1)\phi(\lambda)}}{\sqrt{2\pi(\nu+1)}} \left[\frac{1}{1-\lambda} + \mathrm{O}(\frac{1}{\nu})\right], & (0<\lambda<1) \\
\left(\frac{1}{\sigma-1}\right)^{\nu+1} \nu! \left[\frac{1}{2} + \mathrm{O}(\frac{1}{\sqrt{\nu}})\right], & (\lambda=1) \\
\left(\frac{1}{\sigma-1}\right)^{\nu+1} \nu! \left\{1 - \frac{e^{-(\nu+1)\phi(\lambda)}}{\sqrt{2\pi(\nu+1)}} \left[\frac{1}{\lambda-1} + \mathrm{O}(\frac{1}{\nu})\right]\right\}, & (1<\lambda)
\end{cases}
$$

**証明** 補題 2 に補題 1 を適用する。□

**1.** $1 < XY \ll 1$ **の場合：**

$$X := \exp\left[\frac{a(\nu+1)}{2\nu(\sigma-c)}\right],\ Y := \exp\left[\frac{b(\nu+1)}{2\nu(\sigma-c)}\right],\ 0 < a, b \ll 1,\ c = \frac{1}{2}$$

と置いて補題 4(5) に補題 5 を適用すると

$$\frac{1}{2h}\int_{\gamma_0-h}^{\gamma_0+h}\left|\frac{1+o(1)}{(\sigma-\beta_0)^{\nu+1}}\left\{\alpha+\frac{1}{2\pi i}F(X)\right\}\right|^{\frac{1}{\nu}}dt$$

$$\leq \left[\mathrm{O}\left\{\frac{(XY)^{\nu}}{h}\left(\frac{1}{\nu!}\right)^2\sum_{n\leq(XY)^{\nu}}\frac{\Lambda(n)^2}{n^{2\sigma}}(\log n)^{2\nu}{\alpha_n}^2+\right.\right.$$

$$\left.\left.+\left(Y\frac{X}{U_m}\right)^{2\nu}+\left(\frac{Y}{\sigma}X\right)^{2\nu}\right\}\right]^{\frac{1}{2\nu}}$$

$$=\left[\mathrm{O}\left\{\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(\sigma-\frac{1}{2})}\right]}{h}\left(\frac{1}{\nu!}\right)^2\sum_{n\leq\exp\left[\frac{(a+b)(\nu+1)}{2(\sigma-\frac{1}{2})}\right]}\frac{\Lambda(n)^2}{n^{2\sigma}}(\log n)^{2\nu}{\alpha_n}^2+\right.\right.$$

$$\left.\left.+\left(\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(\sigma-\frac{1}{2})}\right]}{U_m^{\nu}}\right)^2+\left(\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(\sigma-\frac{1}{2})}\right]}{\sigma^{\nu}}\right)^2\right\}\right]^{\frac{1}{2\nu}}$$

$$=\left[\mathrm{O}\left\{\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(\sigma-\frac{1}{2})}\right]}{h}\left(\frac{1}{\nu!}\right)^2(2\nu+2)!\left(\frac{1}{2(\sigma-\frac{1}{2})}\right)^{2\nu+3}\times\right.\right.$$

$$\times\frac{e^{-(2\nu+3)\phi(a+b)}}{\sqrt{2\pi(2\nu+3)}}\frac{1}{1-(a+b)}+$$

$$\left.\left.+\left(\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(\sigma-\frac{1}{2})}\right]}{U_m^{\nu}}\right)^2+\left(\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(\sigma-\frac{1}{2})}\right]}{\sigma^{\nu}}\right)^2\right\}\right]^{\frac{1}{2\nu}}$$

$$=\left[\mathrm{O}\left\{\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(\sigma-\frac{1}{2})}\right]}{h}(2\nu+2)(2\nu+1)\left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu+3}\times\right.\right.$$

$$\times\frac{e^{-(2\nu+3)\phi(a+b)}}{\sqrt{2\pi(2\nu+3)}}\frac{1}{1-(a+b)}+$$

$$+\left(\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(\sigma-\frac{1}{2})}\right]}{U_m^\nu}\right)^2+\left(\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(\sigma-\frac{1}{2})}\right]}{\sigma^\nu}\right)^2\Bigg\}\Bigg]^{\frac{1}{2\nu}}$$

$\zeta(s)=\zeta(\sigma+it)$ の複素零点は $\{s=\sigma+it\in\mathbf{C}\mid 0<\sigma<1\}$ の何処にあるか不明なので $\sigma=1$ と置いて $\nu\to\infty$ とする。$0<a+b\ll 1$ であるのと [1] の補題 4 を使って

$$\begin{aligned}
&\frac{1}{1-\beta_0}\\
&=\lim_{\nu\to\infty}\frac{1}{2h}\int_{\gamma_0-h}^{\gamma_0+h}\left|\frac{1+o(1)}{(1-\beta_0)^{\nu+1}}\left\{\alpha+\frac{1}{2\pi i}F(X)\right\}\right|^{\frac{1}{\nu}}dt\\
&\leq\lim_{\nu\to\infty}\Bigg[\mathrm{O}\Bigg\{\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(1-\frac{1}{2})}\right]}{h}(2\nu+2)(2\nu+1)\left(\frac{1}{1-\frac{1}{2}}\right)^{2\nu+3}\times\\
&\times\frac{e^{-(2\nu+3)\phi(a+b)}}{\sqrt{2\pi(2\nu+3)}}\frac{1}{1-(a+b)}+\\
&+\left(\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(1-\frac{1}{2})}\right]}{U_m^\nu}\right)^2+\left(\frac{\exp\left[\frac{(a+b)(\nu+1)}{2(1-\frac{1}{2})}\right]}{1^\nu}\right)^2\Bigg\}\Bigg]^{\frac{1}{2\nu}}\\
&=\max\left\{\exp\left(\frac{a+b}{2}\right)\frac{1}{1-\frac{1}{2}}e^{-\phi(a+b)},\ \frac{\exp(a+b)}{U_m},\ \exp(a+b)\right\}\\
&=\frac{\exp(a+b)}{U_m}\asymp\log T_m
\end{aligned}$$

即ち

$$1-\beta_0\gg\frac{1}{\log T_m}.$$

これは，1896 年，de la Vallée Poussin が素数定理 Prime Number Theorem を証明したときに同時に得た $\zeta(s)$ の非零領域 zero free region：

$$\left\{s=\sigma+it\mid\sigma>1-\mathrm{O}\left(\frac{1}{\log t}\right)\right\}$$

を意味している。

**2.** $\rho_0 = \beta_0 = 1,\ s = \sigma + it$ **は** $s = 1$ **の近傍の場合：**
次の二つの場合に分かれる：
**2-a.** $\rho_0 = 1$ *and* $\sigma \leq 1$ **の場合と 2-b.** $\rho_0 = 1$ *and* $\sigma > 1$ **の場合**
**2-a.** $\rho_0 = 1$ *and* $\sigma \leq 1$ **の場合:**
この場合は [1] の主定理1の条件：$1=\beta_0 < \sigma$ を満たさないので，考えない。
**2-b.** $\rho_0 = \beta_0 = 1$ *and* $\sigma > 1$ **の場合:**
この場合

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)(\log n)^{\nu}}{n^s}$$

は収束していることに注意する。

$$X := \exp\left[\frac{a(\nu+2)}{\nu(\sigma-1)}\right],\ Y := \exp\left[\frac{b(\nu+2)}{\nu(\sigma-1)}\right],\ \lambda := a+b > 1,\ a, b > 0$$

と置いて補題3, 5を使うと

$$\begin{aligned}
&\left|\frac{1+o(1)}{(\sigma-1)^{\nu+1}}\left\{\alpha + \frac{1}{2\pi i}F(X)\right\}\right| \\
&\leq \left|\frac{1}{\nu!}\sum_{n\leq\exp\left[\frac{(a+b)(\nu+2)}{\sigma-1}\right]} \frac{\Lambda(n)(\log n)^{\nu}}{n^s}\alpha_n\right| + \\
&+\mathrm{O}\left\{\frac{1}{\sigma^{\nu}}\exp\left[\frac{(a+b)(\nu+2)}{\sigma-1}\right]\right\} + \mathrm{O}\left\{\frac{1}{U_{14}^{\nu}}\exp\left[\frac{(a+b)(\nu+2)}{\sigma-1}\right]\right\} \\
&= \mathrm{O}\left\{\frac{1}{\nu!}\sum_{n\leq\exp\left[\frac{(a+b)(\nu+2)}{\sigma-1}\right]} \frac{\Lambda(n)(\log n)^{\nu}}{n^{\sigma}}\right\} + \\
&+\mathrm{O}\left\{\frac{1}{\sigma^{\nu}}\exp\left[\frac{(a+b)(\nu+2)}{\sigma-1}\right]\right\} + \mathrm{O}\left\{\frac{1}{U_{14}^{\nu}}\exp\left[\frac{(a+b)(\nu+2)}{\sigma-1}\right]\right\} \\
&= \mathrm{O}\left\{\frac{1}{\nu!}\left(\frac{1}{\sigma-1}\right)^{\nu+2}(\nu+1)!\left\{1 - \frac{e^{-(\nu+2)\phi(\lambda)}}{\sqrt{2\pi(\nu+2)}}\left[\frac{1}{\lambda-1} + \mathrm{O}(\frac{1}{\nu})\right]\right\}\right\} + \\
&+\mathrm{O}\left\{\frac{1}{\sigma^{\nu}}\exp\left[\frac{\lambda(\nu+2)}{\sigma-1}\right]\right\} + \mathrm{O}\left\{\frac{1}{U_{14}^{\nu}}\exp\left[\frac{\lambda(\nu+2)}{\sigma-1}\right]\right\} \\
&= \mathrm{O}\left\{\left(\frac{1}{\sigma-1}\right)^{\nu+2}(\nu+1)\left\{1 - \frac{e^{-(\nu+2)\phi(\lambda)}}{\sqrt{2\pi(\nu+2)}}\left[\frac{1}{\lambda-1} + \mathrm{O}(\frac{1}{\nu})\right]\right\}\right\} +
\end{aligned}$$

$$+\mathrm{O}\left\{\frac{1}{\sigma^\nu}\exp\left[\frac{\lambda(\nu+2)}{\sigma-1}\right]\right\}+\mathrm{O}\left\{\frac{1}{U_{14}^\nu}\exp\left[\frac{\lambda(\nu+2)}{\sigma-1}\right]\right\}$$

この両辺の $\frac{1}{\nu}$ 乗を取って

$$\left|\frac{1+o(1)}{(\sigma-1)^{\nu+1}}\left\{\alpha+\frac{1}{2\pi i}F(X)\right\}\right|^{\frac{1}{\nu}}$$
$$\leq\left[\mathrm{O}\left\{\left(\frac{1}{\sigma-1}\right)^{\nu+2}(\nu+1)\left\{1-\frac{e^{-(\nu+2)\phi(\lambda)}}{\sqrt{2\pi(\nu+2)}}\left[\frac{1}{\lambda-1}+\mathrm{O}(\frac{1}{\nu})\right]\right\}\right\}+\right.$$
$$\left.+\mathrm{O}\left\{\frac{1}{\sigma^\nu}\exp\left[\frac{\lambda(\nu+2)}{\sigma-1}\right]\right\}+\mathrm{O}\left\{\frac{1}{U_{14}^\nu}\exp\left[\frac{\lambda(\nu+2)}{\sigma-1}\right]\right\}\right]^{\frac{1}{\nu}}$$

ここで $\nu\to\infty$ として，[1] の補題 4 を使うと，

$$\frac{1}{\sigma-1}=\lim_{\nu\to\infty}\left|\frac{1+o(1)}{(\sigma-1)^{\nu+1}}\left\{\alpha+\frac{1}{2\pi i}F(X)\right\}\right|^{\frac{1}{\nu}}$$
$$\leq\lim_{\nu\to\infty}\left[\mathrm{O}\left\{\left(\frac{1}{\sigma-1}\right)^{\nu+2}(\nu+1)\left\{1-\frac{e^{-(\nu+2)\phi(\lambda)}}{\sqrt{2\pi(\nu+2)}}\left[\frac{1}{\lambda-1}+\mathrm{O}(\frac{1}{\nu})\right]\right\}\right\}+\right.$$
$$\left.+\mathrm{O}\left\{\frac{1}{\sigma^\nu}\exp\left[\frac{\lambda(\nu+2)}{\sigma-1}\right]\right\}+\mathrm{O}\left\{\frac{1}{U_{14}^\nu}\exp\left[\frac{\lambda(\nu+2)}{\sigma-1}\right]\right\}\right]^{\frac{1}{\nu}}$$
$$=\max\left\{\frac{1}{\sigma-1},\ \frac{1}{\sigma}\exp\left[\frac{\lambda}{\sigma-1}\right],\ \frac{1}{U_{14}}\exp\left[\frac{\lambda}{\sigma-1}\right]\right\}$$

を得る。これは矛盾のない不等式である。

今度は補題 4 を使うが，補題 4 の中の (3),(4) について考える。

$$XY:=\exp\left[\frac{\lambda_1(2\nu+3)}{\nu(2\sigma-2)}\right]=\exp\left[\frac{\lambda_2(2\nu+3)}{\nu(2\sigma-1)}\right],\ 0<\lambda_1<\lambda_2$$

と置く。素数定理 Prime Number Theorem と補題 2 を使うと

$$\left(\frac{1}{\nu!}\right)^2\sum_{n\leq(XY)^\nu}\frac{\Lambda(n)^2}{n^{2\sigma-1}}(\log n)^{2\nu}{\alpha_n}^2$$
$$<\left(\frac{1}{\nu!}\right)^2(XY)^\nu\sum_{n\leq(XY)^\nu}\frac{\Lambda(n)^2}{n^{2\sigma}}(\log n)^{2\nu}{\alpha_n}^2$$

$$
\begin{aligned}
&\Longleftrightarrow\\
&\left(\frac{1}{\nu!}\right)^2 \sum_{n\le\exp\left[\frac{\lambda_1(2\nu+3)}{2\sigma-2}\right]} \frac{\Lambda(n)^2}{n^{2\sigma-1}}(\log n)^{2\nu}\\
&\ll \left(\frac{1}{\nu!}\right)^2 \exp\left[\frac{\lambda_1(2\nu+3)}{2\sigma-2}\right] \sum_{n\le\exp\left[\frac{\lambda_2(2\nu+3)}{2\sigma-1}\right]} \frac{\Lambda(n)^2}{n^{2\sigma}}(\log n)^{2\nu}\\
&\Longleftrightarrow\\
&\left(\frac{1}{\nu!}\right)^2 \sum_{n\le\exp\left[\frac{\lambda_1(2\nu+3)}{2\sigma-2}\right]} \frac{(\log n)^{2\nu+2}}{n^{2\sigma-1}}\\
&\ll \left(\frac{1}{\nu!}\right)^2 \exp\left[\frac{\lambda_1(2\nu+3)}{2\sigma-2}\right] \sum_{n\le\exp\left[\frac{\lambda_2(2\nu+3)}{2\sigma-1}\right]} \frac{(\log n)^{2\nu+2}}{n^{2\sigma}}\\
&\Longleftrightarrow\\
&\left(\frac{1}{\nu!}\right)^2 \Gamma(2\nu+3)\left(\frac{1}{2\sigma-2}\right)^{2\nu+3} P(2\nu+3,\lambda_1(2\nu+3))\\
&\ll \left(\frac{1}{\nu!}\right)^2 \exp\left[\frac{\lambda_1(2\nu+3)}{2\sigma-2}\right] \Gamma(2\nu+3)\left(\frac{1}{2\sigma-1}\right)^{2\nu+3} P(2\nu+3,\lambda_2(2\nu+3))\\
&\Longleftrightarrow\\
&\left(\frac{1}{\sigma-1}\right)^{2\nu+3} P(2\nu+3,\lambda_1(2\nu+3))\\
&\ll \exp\left[\frac{\lambda_1(2\nu+3)}{2(\sigma-1)}\right]\left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu+3} P(2\nu+3,\lambda_2(2\nu+3))\\
&\quad \mathit{with}\ \ 0<\frac{\lambda_1}{\lambda_2}=\frac{\sigma-1}{\sigma-\frac{1}{2}}<1,\ 1<\sigma
\end{aligned}
$$

となることに注意する。これにも矛盾はない。
**注** $\frac{1}{2}<\sigma\le 1$ となる場合も次のように矛盾はない：

$$
XY := \exp\left[\frac{\lambda(2\nu+3)}{\nu(2\sigma-1)}\right],\ 1<\lambda
$$

と置く。

$$
\begin{aligned}
&\left(\frac{1}{\nu!}\right)^2 \sum_{n\le(XY)^\nu} \frac{\Lambda(n)^2}{n^{2\sigma-1}}(\log n)^{2\nu}{\alpha_n}^2\\
&\asymp \left(\frac{1}{\nu!}\right)^2 \sum_{n\le(XY)^\nu} \frac{(\log n)^{2\nu+2}}{n^{2\sigma-1}}
\end{aligned}
$$

$$
\begin{aligned}
&\asymp \left(\frac{1}{\nu!}\right)^2 \int_1^{(XY)^\nu} \frac{(\log y)^{2\nu+2}}{y^{2\sigma-1}} dy \\
&= \left(\frac{1}{\nu!}\right)^2 \int_0^{\nu\log(XY)} \frac{x^{2\nu+2}}{e^{(2\sigma-1)x}} e^x dx \\
&= \left(\frac{1}{\nu!}\right)^2 \int_0^{\frac{\lambda(2\nu+3)}{2\sigma-1}} x^{2\nu+2} e^{(2-2\sigma)x} dx \\
&\geq \left(\frac{1}{\nu!}\right)^2 \int_0^{\frac{\lambda(2\nu+3)}{2\sigma-1}} x^{2\nu+2} dx \\
&= \left(\frac{1}{\nu!}\right)^2 \frac{1}{2\nu+3} \left[\frac{\lambda(2\nu+3)}{2\sigma-1}\right]^{2\nu+3} \\
&\asymp \frac{e^{2\nu}}{\nu^{2\nu+1}} \frac{1}{2\nu+3} 2^{-(2\nu+3)} \left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu+3} \lambda^{2\nu+3}(2\nu+3)^{2\nu+3} \\
&\asymp (\lambda e)^{2\nu} \frac{1}{\nu} \left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu} \frac{(2\nu+3)^{2\nu}}{(2\nu)^{2\nu}} \\
&\asymp (\lambda e)^{2\nu} \frac{1}{\nu} \left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu} \left(1+\frac{3}{2\nu}\right)^{2\nu} \\
&\asymp (\lambda e)^{2\nu} \frac{1}{\nu} \left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu} e^3 \\
&\asymp (\lambda e)^{2\nu} \frac{1}{\nu} \left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu} \cdots (6)
\end{aligned}
$$

一方,

$$
\begin{aligned}
&\left(\frac{1}{\nu!}\right)^2 (XY)^\nu \sum_{n\leq(XY)^\nu} \frac{\Lambda(n)^2}{n^{2\sigma}} (\log n)^{2\nu} {\alpha_n}^2 \\
&\asymp \left(\frac{1}{\nu!}\right)^2 \exp\left[\frac{\lambda(2\nu+3)}{2\sigma-1}\right] \sum_{n\leq\exp\left[\frac{\lambda_2(2\nu+3)}{2\sigma-1}\right]} \frac{(\log n)^{2\nu+2}}{n^{2\sigma}} \\
&\ll \exp\left[\frac{\lambda(2\nu+3)}{2\sigma-1}\right] \left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu+3} P(2\nu+3, \lambda(2\nu+3)) \\
&\qquad \cdots (7)
\end{aligned}
$$

(6),(7) より

$$(\lambda e)^{2\nu}\frac{1}{\nu}\left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu} \ll \left(\frac{1}{\nu!}\right)^2 \sum_{n\le(XY)^\nu}\frac{\Lambda(n)^2}{n^{2\sigma-1}}(\log n)^{2\nu}{\alpha_n}^2$$

$$< \left(\frac{1}{\nu!}\right)^2 (XY)^\nu \sum_{n\le(XY)^\nu}\frac{\Lambda(n)^2}{n^{2\sigma}}(\log n)^{2\nu}{\alpha_n}^2$$

$$\ll \exp\left[\frac{\lambda(2\nu+3)}{2\sigma-1}\right]\left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu+3} P(2\nu+3,\lambda(2\nu+3))$$

この両辺の $\frac{1}{2\nu}$ 乗をとって，$\nu\to\infty$ とすると，$\lambda>1$ であるから

$$\frac{\lambda e}{\sigma-\frac{1}{2}} \le \lim_{\nu\to\infty}\left[\left(\frac{1}{\nu!}\right)^2 \sum_{n\le(XY)^\nu}\frac{\Lambda(n)^2}{n^{2\sigma-1}}(\log n)^{2\nu}{\alpha_n}^2\right]^{\frac{1}{2\nu}}$$

$$\le \lim_{\nu\to\infty}\left[\left(\frac{1}{\nu!}\right)^2 (XY)^\nu \sum_{n\le(XY)^\nu}\frac{\Lambda(n)^2}{n^{2\sigma}}(\log n)^{2\nu}{\alpha_n}^2\right]^{\frac{1}{2\nu}}$$

$$\le \exp\left[\frac{\lambda}{2\sigma-1}\right]\left(\frac{1}{\sigma-\frac{1}{2}}\right)\lim_{\nu\to\infty}P(2\nu+3,\lambda(2\nu+3))^{\frac{1}{2\nu}}$$

$$= \exp\left[\frac{\lambda}{2\sigma-1}\right]\left(\frac{1}{\sigma-\frac{1}{2}}\right)$$

となり，確かに矛盾は生じない。

$\rho_o=\beta_0+i\gamma_0=1$ として補題4(3) に補題5を適用すると

$$\frac{1}{2h}\int_{-h}^{+h}\left|\frac{1+o(1)}{(\sigma-1)^{\nu+1}}\left\{\alpha+\frac{1}{2\pi i}F(X)\right\}\right|^{\frac{1}{\nu}}dt$$

$$\le \left[\mathrm{O}\left\{\left(\frac{1}{\nu!}\right)^2 \sum_{n\le\exp\left[\frac{\lambda_2(2\nu+3)}{2\sigma-1}\right]}\frac{\Lambda(n)^2}{n^{2\sigma}}(\log n)^{2\nu}{\alpha_n}^2+\right.\right.$$

$$+\mathrm{O}\left(\frac{1}{h}\left(\frac{1}{\nu!}\right)^2 \sum_{n\le\exp\left[\frac{\lambda_1(2\nu+3)}{2\sigma-2}\right]}\frac{\Lambda(n)^2}{n^{2\sigma-1}}(\log n)^{2\nu}{\alpha_n}^2\right)+$$

$$\left.\left.+\left(Y\frac{X}{U_{14}}\right)^{2\nu}+\left(\frac{Y}{\sigma}X\right)^{2\nu}\right\}\right]^{\frac{1}{2\nu}}$$

$$
\begin{aligned}
&\leq \Bigg[ \mathrm{O}\Bigg\{ \left(\frac{1}{\nu!}\right)^2 \sum_{n \leq \exp\left[\frac{\lambda_2(2\nu+3)}{2\sigma-1}\right]} \frac{(\log n)^{2\nu+2}}{n^{2\sigma}} + \\
&+\mathrm{O}\left( \frac{1}{h}\left(\frac{1}{\nu!}\right)^2 \sum_{n \leq \exp\left[\frac{\lambda_1(2\nu+3)}{2\sigma-2}\right]} \frac{(\log n)^{2\nu+2}}{n^{2\sigma-1}} \right) + \\
&+ \left(Y\frac{X}{U_{14}}\right)^{2\nu} + \left(\frac{Y}{\sigma}X\right)^{2\nu} \Bigg\} \Bigg]^{\frac{1}{2\nu}} \\
&\leq \Bigg[ \mathrm{O}\Bigg\{ \left(\frac{1}{\nu!}\right)^2 \left(\frac{1}{2\sigma-1}\right)^{2\nu+3} (2\nu+2)!P(2\nu+3, \lambda_2(2\nu+3)) + \\
&+\mathrm{O}\left( \frac{1}{h}\left(\frac{1}{\nu!}\right)^2 \left(\frac{1}{2\sigma-2}\right)^{2\nu+3} (2\nu+2)!P(2\nu+3, \lambda_1(2\nu+3)) \right) + \\
&+ \left(Y\frac{X}{U_{14}}\right)^{2\nu} + \left(\frac{Y}{\sigma}X\right)^{2\nu} \Bigg\} \Bigg]^{\frac{1}{2\nu}} \\
&\leq \Bigg[ \mathrm{O}\Bigg\{ \left(\frac{1}{\nu!}\right)^2 \frac{(2\nu)!}{2^{2\nu}} \frac{(2\nu+2)(2\nu+1)}{8} \left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu+3} P(2\nu+3, \lambda_2(2\nu+3)) + \\
&+\mathrm{O}\left( \frac{1}{h}\left(\frac{1}{\nu!}\right)^2 \frac{(2\nu)!}{2^{2\nu}} \frac{(2\nu+2)(2\nu+1)}{8} \left(\frac{1}{\sigma-1}\right)^{2\nu+3} P(2\nu+3, \lambda_1(2\nu+3)) \right) + \\
&+ \left(Y\frac{X}{U_{14}}\right)^{2\nu} + \left(\frac{Y}{\sigma}X\right)^{2\nu} \Bigg\} \Bigg]^{\frac{1}{2\nu}} \\
&\leq \Bigg[ \mathrm{O}\Bigg\{ (2\nu+2)(2\nu+1) \left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu+3} P(2\nu+3, \lambda_2(2\nu+3)) + \\
&+\mathrm{O}\left( \frac{1}{h}(2\nu+2)(2\nu+1) \left(\frac{1}{\sigma-1}\right)^{2\nu+3} P(2\nu+3, \lambda_1(2\nu+3)) \right) + \\
&+ \left(Y\frac{X}{U_{14}}\right)^{2\nu} + \left(\frac{Y}{\sigma}X\right)^{2\nu} \Bigg\} \Bigg]^{\frac{1}{2\nu}}
\end{aligned}
$$

$$\le \Bigg[ \mathrm{O}\Bigg\{ (2\nu+2)(2\nu+1)\left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu+3} P(2\nu+3,\lambda_2(2\nu+3)) +$$
$$+\mathrm{O}\left(\nu^2(2\nu+2)(2\nu+1)\left(\frac{1}{\sigma-1}\right)^{2\nu+3} P(2\nu+3,\lambda_1(2\nu+3))\right) +$$
$$+\left(Y\frac{X}{U_{14}}\right)^{2\nu} + \left(\frac{Y}{\sigma}X\right)^{2\nu}\Bigg\}\Bigg]^{\frac{1}{2\nu}}$$

即ち

$$\frac{1}{2h}\int_{-h}^{+h}\left|\frac{1+o(1)}{(\sigma-1)^{\nu+1}}\left\{\alpha+\frac{1}{2\pi i}F(X)\right\}\right|^{\frac{1}{\nu}} dt$$
$$\le \Bigg[ \mathrm{O}\Bigg\{ (2\nu+2)(2\nu+1)\left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu+3} P(2\nu+3,\lambda_2(2\nu+3)) +$$
$$+\mathrm{O}\left(\nu^2(2\nu+2)(2\nu+1)\left(\frac{1}{\sigma-1}\right)^{2\nu+3} P(2\nu+3,\lambda_1(2\nu+3))\right) +$$
$$+\left(Y\frac{X}{U_{14}}\right)^{2\nu} + \left(\frac{Y}{\sigma}X\right)^{2\nu}\Bigg\}\Bigg]^{\frac{1}{2\nu}}$$

を得る。ここで $1<\lambda_1<\lambda_2$ として，$\nu\to\infty$ として [1] の補題4を使うと

$$\frac{1}{\sigma-1}$$
$$=\lim_{\nu\to\infty}\frac{1}{2h}\int_{-h}^{+h}\left|\frac{1+o(1)}{(\sigma-1)^{\nu+1}}\left\{\alpha+\frac{1}{2\pi i}F(X)\right\}\right|^{\frac{1}{\nu}} dt$$
$$\le \lim_{\nu\to\infty}\Bigg[ \mathrm{O}\Bigg\{ (2\nu+2)(2\nu+1)\left(\frac{1}{\sigma-\frac{1}{2}}\right)^{2\nu+3} P(2\nu+3,\lambda_2(2\nu+3)) +$$
$$+\mathrm{O}\left(\nu^2(2\nu+2)(2\nu+1)\left(\frac{1}{\sigma-1}\right)^{2\nu+3} P(2\nu+3,\lambda_1(2\nu+3))\right) +$$
$$+\left(Y\frac{X}{U_m}\right)^{2\nu} + \left(\frac{Y}{\sigma}X\right)^{2\nu}\Bigg\}\Bigg]^{\frac{1}{2\nu}}$$
$$=\max\left\{\frac{1}{\sigma-\frac{1}{2}},\ \frac{1}{\sigma-1},\ \lim_{\nu\to\infty}\frac{XY}{U_{14}},\ \lim_{\nu\to\infty}\frac{XY}{\sigma}\right\}$$

$$= \max \left\{ \frac{1}{\sigma - \frac{1}{2}}, \frac{1}{\sigma - 1}, \frac{\exp\left[\frac{\lambda_2}{\sigma - \frac{1}{2}}\right]}{U_{14}}, \frac{\exp\left[\frac{\lambda_2}{\sigma - \frac{1}{2}}\right]}{\sigma} \right\}$$

即ち $0 < \sigma - 1 \ll 1$ と $\sigma$ を選び固定 fix すれば

$$\begin{aligned} &\frac{1}{\sigma - 1} \\ &\leq \max \left\{ \frac{1}{\sigma - \frac{1}{2}}, \frac{1}{\sigma - 1}, \frac{\exp\left[\frac{\lambda_2}{\sigma - \frac{1}{2}}\right]}{U_{14}}, \frac{\exp\left[\frac{\lambda_2}{\sigma - \frac{1}{2}}\right]}{\sigma} \right\} \\ &= \frac{1}{\sigma - 1} \end{aligned}$$

となって矛盾は生じない。
注　$\frac{1}{2} < \sigma \leq 1$ のとき，この評価 estimation は適用できない。何故ならば，$\frac{1}{2} < \sigma \leq 1$, $m < \nu$, ($m$ は定数 conctant) のとき

$$\left| \sum_{(XY)^\nu} \frac{\Lambda(n)(\log n)^m}{n^{2s-1}} \right| \leq \sum_{(XY)^\nu} \frac{\Lambda(n)(\log n)^m}{n^{2\sigma-1}} \leq \sum_{(XY)^\nu} \frac{\Lambda(n)(\log n)^\nu}{n^{2\sigma-1}}$$

の左辺 left-hand side of the above は，$\nu \to +\infty$ のとき発散 diverge してしまう。

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(received 10 March 2019.)